where α_1, α_2, $\dots$, α_{n-1} are complex numbers (independent of x). Taking $x = 1$, ω, ω^2, $\dots$, ω^{n-1} in succession and adding the results, we obtain

$$\begin{aligned}\sum_{k=0}^{n-1} \det(A + \omega^k B) &= n \det A + \sum_{k=0}^{n-1}\sum_{j=1}^{n-1} \alpha_j \omega^{kj} + \sum_{k=0}^{n-1} \omega^{kn} \det B \\ &= n \det A + \sum_{j=1}^{n-1} \alpha_j \sum_{k=0}^{n-1} \omega^{kj} + \sum_{k=0}^{n-1} \omega^{kn} \det B\,.\end{aligned}$$

Now, since $\omega^n = 1$ and $\sum\limits_{k=0}^{n-1} \omega^{kj} = \dfrac{1-\omega^{nj}}{1-\omega^j} = 0$ for $j = 1, 2, \dots, n-1$, we have

$$\sum_{k=0}^{n-1} \det(A + \omega^k B) = n(\det A + \det B)\,.$$

Then $\sum\limits_{k=0}^{n-1} \det(B + \omega^k A) = n(\det B + \det A)$, and the result follows.

Also solved by PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

3145★. [2006 : 239, 241] *Proposed by Yuming Chen, Wilfrid Laurier University, Waterloo, ON.*

Let $f(x) = x - c^2 \tanh x$, where $c > 1$ is an arbitrary constant. It is not hard to show that $f(x)$ is decreasing on the interval $[-x_0, x_0]$, where $x_0 = \ln\bigl(c + \sqrt{c^2-1}\bigr)$ is the positive root of the equation $\cosh x = c$. For each $x \in (-x_0, x_0)$, the horizontal line passing through $\bigl(x, f(x)\bigr)$ intersects the graph of f at two other points with abscissas $x_1(x)$ and $x_2(x)$. Define a function $g : (-x_0, x_0) \to \mathbb{R}$ as follows:

$$g(x) = x + c^2 \tanh\bigl(x_1(x)\bigr) + c^2 \tanh\bigl(x_2(x)\bigr)\,.$$

Prove or disprove that $g(x) > 0$ for all $x \in (0, x_0)$.

Editor's note: No solutions were received for this problem; hence, it remains open. The proposer believes that the conjecture is true, since there is ample empirical evidence.

3146. [2006 : 239, 242] *Proposed by Vasile Cîrtoaje, University of Ploiesti, Romania.*

Let $p > 1$, and let $a, b, c, d \in \bigl[1/\sqrt{p}, \sqrt{p}\bigr]$. Prove that

(a) $\dfrac{p}{1+p} + \dfrac{2}{1+\sqrt{p}} \le \dfrac{a}{a+b} + \dfrac{b}{b+c} + \dfrac{c}{c+a} \le \dfrac{1}{1+p} + \dfrac{2\sqrt{p}}{1+\sqrt{p}}$;

(b) $\dfrac{p}{1+p} + \dfrac{3}{1+\sqrt[3]{p}} \le \dfrac{a}{a+b} + \dfrac{b}{b+c} + \dfrac{c}{c+d} + \dfrac{d}{d+a} \le \dfrac{1}{1+p} + \dfrac{3\sqrt[3]{p}}{1+\sqrt[3]{p}}$.

Solution by Arkady Alt, San Jose, CA, USA, modified by the editor.

(a) The transposition $(a,b,c) \mapsto (b,a,c)$ in the inequality

$$\frac{a}{a+b}+\frac{b}{b+c}+\frac{c}{c+a} \;\le\; \frac{1}{1+p}+\frac{2\sqrt{p}}{1+\sqrt{p}} \tag{1}$$

gives the equivalent inequality

$$\frac{b}{b+a}+\frac{a}{a+c}+\frac{c}{c+b} \;\le\; \frac{1}{1+p}+\frac{2\sqrt{p}}{1+\sqrt{p}}\,.$$

Since

$$\frac{a}{a+b}+\frac{b}{b+c}+\frac{c}{c+a} \;=\; 3-\left(\frac{b}{b+a}+\frac{a}{a+c}+\frac{c}{c+b}\right)\,,$$

we see that (1) is satisfied if and only if

$$\frac{a}{a+b}+\frac{b}{b+c}+\frac{c}{c+a} \;\ge\; 3-\left(\frac{1}{1+p}+\frac{2\sqrt{p}}{1+\sqrt{p}}\right) \;=\; \frac{p}{1+p}+\frac{2\sqrt{p}}{1+\sqrt{p}}\,.$$

Thus, to prove (a), it is sufficient to prove (1).

Let $x=b/a$, $y=c/b$, and $z=a/c$. Then (1) becomes

$$\frac{1}{1+x}+\frac{1}{1+y}+\frac{1}{1+z} \;\le\; \frac{1}{1+p}+\frac{2\sqrt{p}}{1+\sqrt{p}}\,. \tag{2}$$

Note that $xyz=1$ and $x,\,y,\,z\in[1/p,p]$. To prove (1), it is sufficient to prove (2) for all such x, y, and z.

By the symmetry in (2), we may assume that $z=\max\{x,\,y,\,z\}$. Then, since $xyz=1$ and $z\le p$, we must have $1\le z\le p$ and $1/p\le xy\le 1$. Let $t=\sqrt{xy}$. Then $t^2z=1$ and $1/\sqrt{p}\le t\le 1$. Since $x+y\ge 2\sqrt{xy}=2t$, we have

$$\begin{aligned}\frac{1}{1+x}+\frac{1}{1+y} &= \frac{2+x+y}{1+x+y+xy} = 1+\frac{1-t^2}{1+x+y+t^2}\\ &\le 1+\frac{1-t^2}{1+2t+t^2} = 1+\frac{1-t}{1+t} = \frac{2}{1+t}\,.\end{aligned} \tag{3}$$

Hence,

$$\frac{1}{1+x}+\frac{1}{1+y}+\frac{1}{1+z} \;\le\; \frac{2}{1+t}+\frac{1}{1+z} \;=\; \frac{2}{1+t}+\frac{t^2}{1+t^2}\,.$$

Let $h(t)=\dfrac{2}{1+t}+\dfrac{t^2}{1+t^2}$. Since $h'(t)=\dfrac{-2(1-t)(1-t^3)}{(1+t)^2(1+t^2)^2}$, it follows that h is decreasing on $(0,1]$. Consequently, for $1/\sqrt{p}\le t\le 1$,

$$h(t) \;\le\; h(1/\sqrt{p}) \;=\; \frac{1}{1+p}+\frac{2\sqrt{p}}{1+\sqrt{p}}\,.$$

This proves inequality (2) and completes the proof of (a).

(b) This is treated similarly. The transposition $(a, b, c, d) \mapsto (b, a, d, c)$ in the inequality

$$\frac{a}{a+b} + \frac{b}{b+c} + \frac{c}{c+d} + \frac{d}{d+a} \ \leq\ \frac{1}{1+p} + \frac{3\sqrt[3]{p}}{1+\sqrt[3]{p}} \tag{4}$$

yields the equivalent inequality

$$\frac{b}{b+a} + \frac{a}{a+d} + \frac{d}{d+c} + \frac{c}{c+b} \ \leq\ \frac{1}{1+p} + \frac{3\sqrt[3]{p}}{1+\sqrt[3]{p}}\,.$$

Since

$$\frac{a}{a+b} + \frac{b}{b+c} + \frac{c}{c+d} + \frac{d}{d+a} \ =\ 4 - \left(\frac{b}{b+a} + \frac{a}{a+d} + \frac{d}{d+c} + \frac{c}{c+b}\right),$$

we see that (4) is satisfied if and only if

$$\begin{aligned} \frac{a}{a+b} + \frac{b}{b+c} + \frac{c}{c+d} + \frac{d}{d+a} \ &\geq\ 4 - \left(\frac{p}{p+1} + \frac{3\sqrt[3]{p}}{1+\sqrt[3]{p}}\right) \\ &=\ \frac{p}{p+1} + \frac{3}{1+\sqrt[3]{p}}\,. \end{aligned}$$

Thus, to prove (b), it is sufficient to prove (4).

Let $x = b/a$, $y = c/b$, $u = c/d$, and $v = d/a$. Then (4) becomes

$$\frac{1}{1+x} + \frac{1}{1+y} + \frac{1}{1+u} + \frac{1}{1+v} \ \leq\ \frac{1}{1+p} + \frac{3\sqrt[3]{p}}{1+\sqrt[3]{p}}\,. \tag{5}$$

Note that $xyuv = 1$ and x, y, u, $v \in [1/p, p]$. To prove (4), it is sufficient to prove (5) for all such x, y, u, and v.

Let $t = \sqrt{xy}$ and $s = \sqrt{uv}$. By the symmetry in (5), we may assume that $t \leq s$. Then, since $ts = 1$, we see that $t \leq 1 \leq s$. Furthermore, since $s^2/u = v \leq p$, we have $s^2/p \leq u$, and thus, $s^2/p \leq u \leq p$.

Now, for fixed s,

$$\begin{aligned} &\max\left\{u+v \;\middle|\; uv = s^2,\, \frac{s^2}{p} \leq u \leq p\right\} \\ &\qquad =\ \max\left\{u + \frac{s^2}{u} \;\middle|\; \frac{s^2}{p} \leq u \leq p\right\} \ =\ p + \frac{s^2}{p}\,. \end{aligned}$$

Thus,

$$\begin{aligned} \frac{1}{1+u} + \frac{1}{1+v} \ &=\ \frac{2+u+v}{1+u+v+uv} \ =\ 1 - \frac{s^2-1}{1+u+v+s^2} \\ &\leq\ 1 - \frac{s^2-1}{1+p+\dfrac{s^2}{p}+s^2} \ =\ \frac{2+p+\dfrac{s^2}{p}}{1+p+\dfrac{s^2}{p}+s^2} \\ &=\ \frac{p}{s^2+p} + \frac{1}{1+p}\,. \end{aligned} \tag{6}$$

Using inequalities (6) and (3), we get

$$\frac{1}{1+x}+\frac{1}{1+y}+\frac{1}{1+u}+\frac{1}{1+v} \;\le\; \frac{2s}{1+s}+\frac{p}{s^2+p}+\frac{1}{1+p}\,.$$

Let $g(s) = \frac{2s}{1+s}+\frac{p}{s^2+p}$. Since $g'(s) = \frac{2(s-p)(s^3-p)}{(s+1)^2(s^2+p)^2}$, this function has a local maximum at $s = \sqrt[3]{p}$, which is in the interval $(1,p)$. We have $g(1) = -1+\frac{p}{1+p} < 0$ and $g(p) = -\frac{2}{p+1}+\frac{1}{p+1} < 0$; whence, $\max\limits_{s\in[1,p]} g(s) = g\big(\sqrt[3]{p}\big)$, and therefore,

$$\begin{aligned}\frac{1}{1+x}+\frac{1}{1+y}+\frac{1}{1+u}+\frac{1}{1+v} \;&\le\; \frac{2\sqrt[3]{p}}{1+\sqrt[3]{p}}+\frac{p}{\sqrt[3]{p^2}+p}+\frac{1}{1+p}\\ &=\; \frac{3\sqrt[3]{p}}{1+\sqrt[3]{p}}+\frac{1}{1+p}\,.\end{aligned}$$

This proves (4) and completes the proof of (b).

Also solved by WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria (part (a)); PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

3147. [2006 : 239, 242] *Proposed by Vasile Cîrtoaje, University of Ploiesti, Romania; and Gabriel Dospinescu, Paris, France.*

Let $n \ge 3$, and let $x_1, x_2, \ldots, x_n$ be positive real numbers such that $x_1x_2\cdots x_n = 1$. For $n = 3$ and $n = 4$, prove that

$$\frac{1}{x_1^2+x_1x_2}+\frac{1}{x_2^2+x_2x_3}+\cdots+\frac{1}{x_n^2+x_nx_1} \;\ge\; \frac{n}{2}\,.$$

Solution by the proposer.

Using the substitutions $x_1 = \sqrt{\frac{a_2}{a_1}}$, $x_2 = \sqrt{\frac{a_3}{a_2}}$, $\ldots$, $x_n = \sqrt{\frac{a_1}{a_n}}$, the given inequality becomes

$$\frac{a_1}{a_2+\sqrt{a_1a_3}}+\frac{a_2}{a_3+\sqrt{a_2a_4}}+\cdots+\frac{a_n}{a_2+\sqrt{a_na_2}} \;\ge\; \frac{n}{2}\,.$$

Since $\sqrt{a_1a_3} \le \frac{a_1+a_3}{2}$, $\ldots$, $\sqrt{a_na_2} \le \frac{a_n+a_2}{2}$, it suffices to show that

$$\frac{a_1}{a_1+2a_2+a_3}+\frac{a_2}{a_2+2a_3+a_4}+\cdots+\frac{a_n}{a_n+2a_1+a_2} \;\ge\; \frac{n}{4}\,.$$

By the Cauchy-Schwarz Inequality, we have

$$\begin{aligned}(a_1+\cdots+a_n)^2 \;&\le\; [a_1(a_1+2a_2+a_3)+\cdots+a_n(a_n+2a_1+a_2)]\\ &\qquad\cdot\left(\frac{a_1}{a_1+2a_2+a_3}+\cdots+\frac{a_n}{a_n+2a_1+a_2}\right)\,.\end{aligned}$$